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Convergence of Hermite–Fejér type interpolation of higher order on an arbitrary system of nodes ☆

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Abstract

A criterion of convergence for general Hermite–Fejér type interpolation of higher order on an arbitrary system of nodes is given.

1. Introduction

Let $\mathbb{N}_1 := \{1, 3, 5, \dots\}$ and $\mathbb{N}_2 := \{2, 4, 6, \dots\}$. Let $n \in \mathbb{N} (n \geqslant 2)$, $m_{kn} \in \mathbb{N}$ $(k = 1, 2, \dots, n,\ n = 2, 3, \dots)$, and

$$x_{0n} := 1 \geqslant x_{1n} > x_{2n} > \cdots > x_{nn} \geqslant x_{n+1,n} := -1, \quad n = 2, 3, \dots \ . \tag{1.1}$$

In what follows we shall often omit the superfluous notations, i.e., $m_{kn}, x_{kn}, \dots$ will be denoted by $m_k, x_k, \dots$, etc. Throughout this paper let $N := N_n := \sum_{k=1}^{n} m_{kn} - 1$ and $m := \max_{1 \leqslant k \leqslant n,\ n \geqslant 2} m_{kn} < +\infty$. Denote by $\mathbf{P}_N$ the set of polynomials of degree at most N and by A_{jk} the fundamental polynomials for Hermite interpolation, i.e., $A_{jk} \in \mathbf{P}_N$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \quad p = 0, 1, \dots, m_q - 1, \quad j = 0, 1, \dots, m_k - 1,$$
$$q, k = 1, 2, \dots, n. \tag{1.2}$$

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The Hermite–Fejér type interpolation for $f \in C[-1,1]$ is given by

$$H_{nm}(f,x) = \sum_{k=1}^{n} f(x_k)A_{0k}(x). \tag{1.3}$$

In [5] the author established a criterion of convergence for Hermite–Fejér type interpolation of higher order on an arbitrary system of nodes as follows ($\|\cdot\|$ stands for the uniform norm on $[-1,1]$ and $f_i(x) := x^i,\ i = 1,2,\dots$).

Theorem A (Shi [5, Theorem 4.2]). *Let* $m_{kn} \equiv m \in \mathbb{N}_2$. *Then*

$$\lim_{n\to\infty} \|H_{nm}(f) - f\| = 0 \tag{1.4}$$

holds for all $f \in C[-1,1]$ *if and only if*

$$\|H_{nm}\| := \left\| \sum_{k=1}^{n} |A_{0k}| \right\| = O(1) \tag{1.5}$$

is true and (1.4) *holds for the two monomials* $f = f_i,\ i = 1,2$.

The main aim of this paper is to establish a criterion of convergence for general Hermite–Fejér type interpolation of higher order on an arbitrary system of nodes, replacing the assumption that $m_{kn} \equiv m \in \mathbb{N}_2$ by the assumption that all $m_{kn} \in \mathbb{N}_2$. That is the following

Theorem. *Let all* $m_{kn} \in \mathbb{N}_2$. *Then relation* (1.4) *holds for all* $f \in C[-1,1]$ *if and only if relation* (1.5) *is true and* (1.4) *holds for the two monomials* $f = f_i,\ i = 1,2$.

This extension is not trivial. To prove the theorem we have to prove a series of auxillary lemmas, which are of independent interest and put in the next section. Then we give the proof of the theorem in the last section.

2. Auxiliary lemmas

First we state some known results needed later.

Lemma A (Borwein and Erdélyi [1, p. 235]). *Let* $P \in \mathbf{P}_n$. *Then*

$$|P(y)| \leqslant |T_n(y)|\,\|P\| = \frac{1}{2}\{[y + (y^2-1)^{1/2}]^n + [y - (y^2-1)^{1/2}]^n\}\|P\|, \quad |y| > 1, \tag{2.1}$$

where T_n *stands for the nth Chebyshev polynomial of the first kind.*

Lemma B (Shi [5, Lemma 4.1]). *Let* $P_k \in \mathbf{P}_n$, $k = 1, 2, \dots, M$, *and* $1 \geqslant y_1 > y_2 > \cdots > y_M \geqslant -1$. *If*

$$\left\| \sum_{k=1}^{M} |(x - y_k) P_k(x)| \right\| = \mu_n$$

and

$$\sum_{k=1}^{M} |P_k(y_j)| \leqslant v_n, \quad j = 1, 2, \dots, M,$$

then

$$\left\| \sum_{k=1}^{M} |P_k| \right\| \leqslant 2(n^2 \mu_n + v_n).$$

In particular, if $M = 1$ *and* $P_1(y_1) = 0$, $|y_1| < 1$, *then*

$$\|P_1\| \leqslant \frac{4n\mu_n}{(1 - y_1^2)^{1/2}}.$$

Lemma C (Shi [5, Theorem 2.1]). *If for a fixed* n, $m_k - j$ *is odd and* $j < i \leqslant m_k - 1$ *then*

$$|A_{ik}(x)| \leqslant \frac{j!}{i!} d_k^{i-j} |A_{jk}(x)|, \quad x \in [-1, 1], \quad 1 \leqslant k \leqslant n, \tag{2.2}$$

where $d_k \coloneqq \max\{|x_k - x_{k-1}|, |x_k - x_{k+1}|\}$, $k = 1, 2, \dots, n$.

Lemma 1. *Let all* $m_{kn} \in \mathbb{N}_2$. *If*

$$\|H_{nm}\| = \mu_n, \tag{2.3}$$

then

$$\left\| \sum_{k=1}^{n} |A_{1k}| \right\| \leqslant 8m^2 n^2 \mu_n. \tag{2.4}$$

Proof. By the same argument as that of [5, Theorem 2.3] we can get

$$\sum_{k=1}^{n} (x - x_k) A_{1k}(x) = \sum_{k=1}^{n} |(x - x_k) A_{1k}(x)| \leqslant \sum_{k=1}^{n} (x - x_k)^2 A_{0k}(x), \quad x \in \mathbb{R}. \tag{2.5}$$

Hence by (2.3) we obtain

$$\sum_{k=1}^{n} (x - x_k) A_{1k}(x) \leqslant 4\mu_n,$$

which according to Lemma B gives (2.4) (by $\deg A_{ik} < nm$). □

Lemma 2. *Let A^*_{jk} be the fundamental polynomials for Hermite interpolation based on the system of nodes*

$$x^*_{kn} = ax_{kn}, \quad 0<a<1, \quad k = 1,2,\dots,n, \quad n = 2,3,\dots. \tag{2.6}$$

Then with the notations $x = \cos\theta$ and $x^* = \cos\theta^*$

$$\theta^*_{kn} = \arccos(a\cos\theta_{kn}), \quad k = 1,2,\dots,n, \tag{2.7}$$

$$A^*_{jk}(x) = a^j A_{jk}(x/a), \quad k = 1,2,\dots,n, \quad j = 0,1,\dots,m_k - 1, \tag{2.8}$$

$$\max_{|x|\leqslant 1} |A^*_{jk}(x)| = a^j \max_{|x|\leqslant 1/a} |A_{jk}(x)|, \quad k = 1,2,\dots,n, \quad j = 0,1,\dots,m_k - 1, \tag{2.9}$$

and

$$||H^*_{nm}|| := \left|\left|\sum_{k=1}^{n} |A^*_{0k}|\right|\right| \leqslant [a^{-1} + (a^{-2} - 1)^{1/2}]^n ||H_{nm}||. \tag{2.10}$$

Proof. Relations (2.7)–(2.9) may be obtained directly from the definition. Now let us prove (2.10). Assume that

$$\sum_{k=1}^{n} |A_{0k}(y)| = \max_{|x|\leqslant 1/a} \sum_{k=1}^{n} |A_{0k}(x)|, \quad |y|\leqslant 1/a.$$

This relation, together with (2.8), yields

$$\begin{aligned} ||H^*_{nm}|| &= \max_{|x|\leqslant 1} \sum_{k=1}^{n} |A^*_{0k}(x)| = \max_{|x|\leqslant 1} \sum_{k=1}^{n} |A_{0k}(x/a)| \\ &= \max_{|x|\leqslant 1/a} \sum_{k=1}^{n} |A_{0k}(x)| = \sum_{k=1}^{n} |A_{0k}(y)|. \end{aligned} \tag{2.11}$$

If $|y|\leqslant 1$ then by (2.11) we have that

$$||H^*_{nm}|| = \sum_{k=1}^{n} |A_{0k}(y)| \leqslant ||H_{nm}||;$$

if $|y|>1$ then by (2.11) and (2.1) we have that

$$\begin{aligned} ||H^*_{nm}|| &= \sum_{k=1}^{n} |A_{0k}(y)| = \sum_{k=1}^{n} [\operatorname{sgn} A_{0k}(y)]A_{0k}(y) \\ &\leqslant \frac{1}{2}\{[y + (y^2 - 1)^{1/2}]^n + [y - (y^2 - 1)^{1/2}]^n\} \left|\left|\sum_{k=1}^{n} [\operatorname{sgn} A_{0k}(y)]A_{0k}\right|\right| \\ &\leqslant [|y| + (y^2 - 1)^{1/2}]^n ||H_{nm}|| \\ &\leqslant [a^{-1} + (a^{-2} - 1)^{1/2}]^n ||H_{nm}||. \qquad \square \end{aligned}$$

In the sequel $c, c', \ldots$ will stand for positive constants depending only on m, unless otherwise indicated; their values may be different at different occurrences, even in subsequent formulas; in addition, $c_r, c_r' \geqslant 1$ will stand for positive constants depending only on r and increasing with respect to r, their values may also be different at different occurrences, even in subsequent formulas.

Lemma 3. *Let* $r \in \mathbb{N}$, $p > 0$, *and*

$$f(\theta) = (\sin p\theta)^r. \tag{2.12}$$

Then,

$$|f^{(j)}(\theta)| \leqslant c_j (rp)^j |\sin p\theta|^{r-j}, \quad 0 \leqslant j \leqslant r. \tag{2.13}$$

Proof (By induction). Relation (2.13) with $j = 0$ is trivial. Suppose now, as an induction hypothesis, that relation (2.13) holds for all j, $j \leqslant q < r$. Differentiating (2.12) yields

$$f'(\theta) = rp(\sin p\theta)^{r-1} \cos p\theta$$

and then differentiating the above relation q times gives (using the induction hypothesis)

$$\begin{aligned}
|f^{(q+1)}(\theta)| &= rp \left| \sum_{j=0}^{q} \binom{q}{j} [(\sin p\theta)^{r-1}]^{(j)} (\cos p\theta)^{(q-j)} \right| \\
&\leqslant rp \sum_{j=0}^{q} \binom{q}{j} c_j [(r-1)p]^j |\sin p\theta|^{r-1-j} p^{q-j} \\
&\leqslant c_q (rp)^{q+1} |\sin p\theta|^{r-1-q} \sum_{j=0}^{q} \binom{q}{j} \\
&= c_q (rp)^{q+1} |\sin p\theta|^{r-1-q} 2^q \\
&= c_{q+1} (rp)^{q+1} |\sin p\theta|^{r-(q+1)},
\end{aligned}$$

which shows that relation (2.13) is true for $j = q + 1$. By induction this proves (2.13). □

Lemma 4. *Let* $r \in \mathbb{N}$, $p > 0$, *and*

$$g(\theta) := \left[\frac{\sin(p\theta/2)}{\sin(\theta/2)} \right]^r. \tag{2.14}$$

Then

$$|g^{(j)}(\theta)| \leqslant c_j (rp)^j |\sin(\theta/2)|^{-r-j}, \quad \theta \neq 0, \quad 0 \leqslant j \leqslant r. \tag{2.15}$$

Proof (By induction). Relation (2.14) implies

$$|g(\theta)| \leqslant |\sin(\theta/2)|^{-r},$$

which implies that relation (2.15) is true for $j = 0$. Suppose now, as an induction hypothesis, that relation (2.15) is true for all $j,\ j \leqslant q < r$. Rewrite (2.14) as

$$g(\theta)[\sin(\theta/2)]^r = [\sin(p\theta/2)]^r.$$

Differentiating this relation $(q+1)$ times and then using (2.13) (replacing c_j by $c_j{}'$ in it) and the induction hypothesis, we get

$$\begin{aligned}
&|g^{(q+1)}(\theta)[\sin(\theta/2)]^r| \\
&\quad = \left| [\sin(p\theta/2)]^r]^{(q+1)} - \sum_{j=0}^{q} \binom{q+1}{j} g^{(j)}(\theta)[\sin(\theta/2)]^r]^{(q+1-j)} \right| \\
&\quad \leqslant c_{q+1}{}'(rp/2)^{q+1}|\sin(p\theta/2)|^{r-q-1} \\
&\qquad + \sum_{j=0}^{q} \binom{q+1}{j} c_j (rp)^j |\sin(\theta/2)|^{-r-j} c_{q+1-j}{}'(r/2)^{q+1-j} |\sin(\theta/2)|^{r-(q+1-j)} \\
&\quad \leqslant c_{q+1}{}'(rp)^{q+1} + c_q c_{q+1}{}'(rp)^{q+1}|\sin(\theta/2)|^{-q-1} \sum_{j=0}^{q} \binom{q+1}{j} \\
&\quad \leqslant c_{q+1}(rp)^{q+1}|\sin(\theta/2)|^{-q-1} \sum_{j=0}^{q+1} \binom{q+1}{j} \\
&\quad \leqslant c_{q+1}(rp)^{q+1}|\sin(\theta/2)|^{-q-1}.
\end{aligned}$$

Then

$$|g^{(q+1)}(\theta)| \leqslant c_{q+1}(rp)^{q+1}|\sin(\theta/2)|^{-r-(q+1)},$$

which shows that relation (2.15) is true for $j = q+1$. By induction this proves (2.15). □

Lemma 5. *Let* $\psi \in \mathbf{P}_n$ *and*

$$\phi(\theta) = \psi(\cos\theta), \quad 0 \leqslant \theta \leqslant \pi. \tag{2.16}$$

Then,

$$|\psi^{(j)}(\cos\theta)| \leqslant \frac{c_j}{(\sin\theta)^{j^2}} \sum_{v=1}^{j} |\phi^{(v)}(\theta)| \ (\sin\theta \neq 0), \quad j \geqslant 1. \tag{2.17}$$

Proof. First let us prove that

$$\phi^{(j)}(\theta) = \sum_{v=1}^{j} c_{vj}(\theta)\psi^{(v)}(\cos\theta), \quad j \geqslant 1, \tag{2.18}$$

where $c_{vj}(\theta)$ is a trigonometric polynomial of degree v and

$$c_{jj}(\theta) = (-\sin\theta)^j, \quad |c_{vj}(\theta)| \leqslant j!, \quad v = 1, 2, \ldots, j. \tag{2.19}$$

We use induction. It is easy to see that relations (2.18) and (2.19) hold for $j = 1$. Suppose now, as an induction hypothesis, that relations (2.18) and (2.19) are true for all $j,\ 1 \leqslant j \leqslant q$. By differentiation of (2.18) with $j = q$ we obtain

$$\begin{aligned}\phi^{(q+1)}(\theta) &= \sum_{v=1}^{q} [c_{vq}{}'(\theta)\psi^{(v)}(\cos\theta) - c_{vq}(\theta)\psi^{(v+1)}(\cos\theta)\sin\theta] \\ &= \sum_{v=1}^{q+1} [c_{vq}{}'(\theta) - c_{v-1,q}(\theta)\sin\theta]\psi^{(v)}(\cos\theta) \\ &= \sum_{v=1}^{q+1} c_{v,q+1}(\theta)\psi^{(v)}(\cos\theta),\end{aligned}$$

where $c_{0,q}(\theta) = c_{q+1,q}(\theta) = 0$ and

$$c_{v,q+1}(\theta) = c_{vq}{}'(\theta) - c_{v-1,q}(\theta)\sin\theta. \tag{2.20}$$

Hence by (2.19) $c_{q+1,q+1}(\theta) = -c_{qq}(\theta)\sin\theta = (-\sin\theta)^{q+1}$ and $||c_{q+1,q+1}|| \leqslant 1$ (here $||\cdot||$ stads for the uniform norm on $[0, 2\pi]$); by Bernstein inequality it follows from (2.20) that $||c_{vq}{}'|| \leqslant v||c_{vq}||$ and hence

$$||c_{v,q+1}|| \leqslant v||c_{vq}|| + ||c_{v-1,q}|| \leqslant (v+1)q! \leqslant (q+1)!, \quad 1 \leqslant v \leqslant q.$$

This shows that relations (2.18) and (2.19) are true for $j = q + 1$. By induction this proves (2.18) and (2.19).

For the proof of (2.17) we again use induction. It is easy to see that relation (2.17) is true for $j = 1$. Suppose now, as an induction hypothesis, that relation (2.17) is true for all $j,\ 1 \leqslant j \leqslant q$. It follows from (2.18), (2.19), and the induction hypothesis that

$$\begin{aligned}|\psi^{(q+1)}(\cos\theta)| &\leqslant \frac{1}{(\sin\theta)^{q+1}}\left[|\phi^{(q+1)}(\theta)| + (q+1)! \sum_{j=1}^{q} |\psi^{(j)}(\cos\theta)|\right] \\ &\leqslant \frac{1}{(\sin\theta)^{q+1}}\left[|\phi^{(q+1)}(\theta)| + (q+1)! \sum_{j=1}^{q} \frac{c_j}{(\sin\theta)^{j^2}} \sum_{v=1}^{j} |\phi^{(v)}(\theta)|\right] \\ &\leqslant \frac{1}{(\sin\theta)^{q+1}}\left[|\phi^{(q+1)}(\theta)| + \frac{(q+1)!c_q}{(\sin\theta)^{q^2}} \sum_{j=1}^{q} \sum_{v=1}^{j} |\phi^{(v)}(\theta)|\right] \\ &\leqslant \frac{1}{(\sin\theta)^{q+1}}\left[|\phi^{(q+1)}(\theta)| + \frac{q(q+1)!c_q}{(\sin\theta)^{q^2}} \sum_{j=1}^{q} |\phi^{(j)}(\theta)|\right] \\ &\leqslant \frac{c_{q+1}}{(\sin\theta)^{(q+1)^2}} \sum_{j=1}^{q+1} |\phi^{(j)}(\theta)|.\end{aligned}$$

This shows that relation (2.17) is true for $j = q + 1$ and by induction proves (2.17). □

Lemma 6. *Let* $0 \leqslant \theta, \tau \leqslant \pi$, *g be defined by* (2.14), *and*

$$\phi(\theta) = p^{-r}[g(\theta+\tau) + g(\theta-\tau)]. \tag{2.21}$$

Let ψ *be defined by* (2.16). *Then*

$$|\psi^{(j)}(\cos\theta)| \leqslant \frac{c_j(rp)^j}{p^r(\sin\theta)^{j^2}}\left|\frac{3\pi}{\theta-\tau}\right|^{r+j}, \quad 0<\theta<\pi, \quad \theta\neq\tau, \quad 0\leqslant j\leqslant r. \tag{2.22}$$

Proof. Suppose without loss of generality that $0<\theta\leqslant\pi/2$. By (2.21) and (2.15)

$$|\phi^{(j)}(\theta)| \leqslant \frac{c_j(rp)^j}{p^r}\left\{\frac{1}{|\sin[(\theta+\tau)/2]|^{r+j}} + \frac{1}{|\sin[(\theta-\tau)/2]|^{r+j}}\right\}. \tag{2.23}$$

Since $0<\theta\leqslant\pi/2$, we have $0<(\theta+\tau)/2\leqslant 3\pi/4$ and hence

$$\sin\frac{\theta+\tau}{2} \geqslant \begin{cases} \frac{2}{\pi}\left(\frac{\theta+\tau}{2}\right) = \frac{\theta+\tau}{\pi}, & \frac{\theta+\tau}{2}\leqslant\frac{\pi}{2}, \\ 2^{-1/2} \geqslant 2^{-1/2}\frac{2}{\pi}\left(\frac{\theta+\tau}{2}\right) \geqslant \frac{\theta+\tau}{2\pi}, & \frac{\pi}{2}<\frac{\theta+\tau}{2}\leqslant\frac{3\pi}{4}. \end{cases} \tag{2.24}$$

This implies

$$\sin\frac{\theta+\tau}{2} \geqslant \frac{\theta+\tau}{2\pi}, \quad 0<\frac{\theta+\tau}{2}\leqslant\frac{3\pi}{4}. \tag{2.25}$$

Of course, we have

$$\left|\sin\frac{\theta-\tau}{2}\right| \geqslant \left|\frac{\theta-\tau}{\pi}\right|.$$

Substituting the above inequalities into (2.23), we obtain

$$\begin{aligned} |\phi^{(j)}(\theta)| &\leqslant \frac{c_j(rp)^j}{p^r}\left\{\left|\frac{2\pi}{\theta+\tau}\right|^{r+j} + \left|\frac{\pi}{\theta-\tau}\right|^{r+j}\right\} \\ &\leqslant \frac{c_j(rp)^j}{p^r}\left|\frac{3\pi}{\theta-\tau}\right|^{r+j}, \quad j\geqslant 0. \end{aligned} \tag{2.26}$$

Inserting estimation (2.26) into (2.17), we have

$$\begin{aligned} |\psi^{(j)}(\cos\theta)| &\leqslant \frac{c_j}{(\sin\theta)^{j^2}}\sum_{v=1}^{j}|\phi^{(v)}(\theta)| \\ &\leqslant \frac{c_j}{(\sin\theta)^{j^2}}\sum_{v=1}^{j}\frac{c_v(rp)^v}{p^r}\left|\frac{3\pi}{\theta-\tau}\right|^{r+v} \\ &\leqslant \frac{c_j(rp)^j}{p^r(\sin\theta)^{j^2}}\left|\frac{3\pi}{\theta-\tau}\right|^{r+j}. \qquad \square \end{aligned}$$

Lemma 7. *Let all* $m_{kn}\in\mathbb{N}_1$ *or all* $m_{kn}\in\mathbb{N}_2$. *If*

$$||H_{nm}|| = \mu_n, \tag{2.27}$$

then

$$\theta_{k+1,n} - \theta_{k,n} \leqslant \frac{c\ln(n\mu_n)}{n}, \quad k = 0, 1, \ldots, n, \quad n = 2, 3, \ldots \quad (2.28)$$

Proof. The proof follows the line given by Erdős and Turán [3, p. 718] and [2,4]. Let

$$n \geqslant \ln(n^{m^2+4}\mu_n) + 2, \quad (2.29)$$

$r \in \mathbb{N}_2$, and $p \in \mathbb{N}_1 (p \geqslant 3)$ such that

$$\frac{1}{2} r(p-1) \leqslant n-1. \quad (2.30)$$

Suppose that

$$\max_{0 \leqslant k \leqslant n} (\theta_{k+1,n} - \theta_{k,n}) = \theta_{i+1,n} - \theta_{i,n} := 2\delta_n, \quad \tau := \frac{1}{2}(\theta_{i+1,n} + \theta_{i,n}),$$
$$\xi := \cos\tau. \quad (2.31)$$

Let ϕ and ψ be defined by (2.21) and (2.16), respectively. Observing that $\psi(\cdot) \in \mathbf{P}_{n-1}$, we see

$$\phi(\theta) = \psi(\cos\theta) = \psi(x) = \sum_{k=1}^{n} \sum_{j=0}^{m_k-1} \psi^{(j)}(x_k) A_{jk}(x). \quad (2.32)$$

If all $m_{kn} \in \mathbb{N}_2$, then we have by (2.4) and (2.2)

$$||A_{jk}|| \leqslant \frac{1}{j!} d_k^{j-1} ||A_{1k}|| \leqslant 8m^2 n^2 \mu_n, \quad j \geqslant 1, \quad 1 \leqslant k \leqslant n,$$

which, coupled with (2.27), gives

$$||A_{jk}|| \leqslant 8m^2 n^2 \mu_n, \quad j \geqslant 0, \quad 1 \leqslant k \leqslant n. \quad (2.33)$$

If all $m_{kn} \in \mathbb{N}_1$, then we have by (2.2)

$$||A_{jk}|| \leqslant \frac{1}{j!} d_k^{j} ||A_{0k}|| \leqslant 2\mu_n,$$

which also implies (2.33).

We separate two cases when $\min\{\theta_1, \pi - \theta_n\} \geqslant 1/n$ and when $\min\{\theta_1, \pi - \theta_n\} < 1/n$.

Case 1: $\min\{\theta_1, \pi - \theta_n\} \geqslant 1/n$. In [4, p. 537] it is proved that $\phi(\tau) \geqslant 1$. Using this inequality, and applying (2.31)–(2.33) and (2.22), we obtain

$$\begin{aligned}
1 \leqslant \phi(\tau) &= \sum_{k=1}^{n} \sum_{j=0}^{m_k-1} \psi^{(j)}(x_k) A_{jk}(\xi) \\
&\leqslant \sum_{k=1}^{n} \sum_{j=0}^{m_k-1} \frac{c_j (rp)^j}{p^r (\sin \theta_k)^{j^2}} \left| \frac{3\pi}{\theta_k - \tau} \right|^{r+j} 8m^2 n^2 \mu_n \\
&\leqslant \frac{c_m (rp)^m n^2 \mu_n}{p^r} \left(\frac{3\pi}{\delta_n} \right)^{r+m} \sum_{k=1}^{n} \sum_{j=0}^{m_k-1} \frac{1}{(\sin \theta_k)^{j^2}} \\
&\leqslant \frac{c_m (rp)^m n^{(m-1)^2+3} \mu_n}{p^r} \left(\frac{3\pi}{\delta_n} \right)^{r+m}.
\end{aligned}$$

Since the inequality $p \geqslant 3$ implies that $p - 1 \geqslant 2p/3$, by (2.30) we get $rp \leqslant 3n$. Using this inequality and observing that $\delta_n \geqslant \pi/[2(n+1)]$, we get

$$1 \leqslant c_m n^{m^2+4} \mu_n \left(\frac{3\pi}{p\delta_n} \right)^r$$

and hence

$$\delta_n \leqslant \frac{3\pi}{p} [c_m n^{m^2+4} \mu_n]^{1/r}. \tag{2.34}$$

Now choose

$$r = 2\left[\frac{1}{2} \ln(n^{m^2+4} \mu_n) \right]$$

and

$$p = 1 + 2\left[\frac{n-1}{r} \right],$$

which obviously satisfy condition (2.30). From these definitions we can obtain estimations

$$r \leqslant \ln(n^{m^2+4} \mu_n) \tag{2.35}$$

and

$$r \geqslant 2\left\{ \frac{1}{2} \ln(n^{m^2+4} \mu_n) - 1 \right\} \geqslant \frac{1}{2} \ln(n^{m^2+4} \mu_n), \tag{2.36}$$

because $m \geqslant 2$, $n \geqslant 2$, and $\mu_n \geqslant 1$; meanwhile, by (2.29) and (2.35)

$$p \geqslant 1 + 2\left(\frac{n-1}{r} - 1 \right) = \frac{2(n-1) - r}{r} \geqslant \frac{n}{\ln(n^{m^2+4} \mu_n)}. \tag{2.37}$$

Inserting estimations (2.36) and (2.37) into (2.34) we see

$$\delta_n \leqslant \frac{3\pi \ln(n^{m^2+4} \mu_n)}{n} \{c_m n^{m^2+4} \mu_n\}^{2/\ln(n^{m^2+4} \mu_n)}. \tag{2.38}$$

It is clear that for $x>0$ and $x\neq 1$

$$x^{1/\ln x} = e^{\ln(x^{1/\ln x})} = e.$$

Using this identity inequality (2.38) becomes

$$\delta_n \leqslant \frac{3\pi \ln(n^{m^2+4}\mu_n)e^2}{n} c_m^{2/\ln(n^{m^2+4}\mu_n)} \leqslant \frac{c\ln(n\mu_n)}{n}. \tag{2.39}$$

Case 2: $\min\{\theta_1, \pi-\theta_n\}<1/n$. In this case let us consider the Hermite interpolation based on the system of nodes (2.6) with

$$a = \frac{n^2}{n^2+1}. \tag{2.40}$$

By (2.10)

$$||H_{nm}^*|| \leqslant \left(\frac{n^2+1}{n^2} + \frac{2n+1}{n^2}\right)^n \mu_n \leqslant \left(1+\frac{3}{n}\right)^n \mu_n \leqslant c\mu_n. \tag{2.41}$$

By means of (2.6) we see that

$$a \geqslant a\cos\theta_1 = \cos\theta_1^* = 1 - 2\sin^2\frac{\theta_1^*}{2} \geqslant 1 - \frac{(\theta_1^*)^2}{2}$$

and hence according to (2.40)

$$\theta_1^* \geqslant [2(1-a)]^{1/2} \geqslant \frac{1}{n}. \tag{2.42}$$

Similarly, we can conclude that

$$\pi - \theta_n^* \geqslant \frac{1}{n}. \tag{2.43}$$

Then we can apply the conclusion of Case 1 to the Hermite interpolation based on the system of nodes (2.6) with (2.40) to obtain (using (2.41))

$$\theta_{k+1,n}^* - \theta_{k,n}^* \leqslant \frac{c\ln(n\mu_n)}{n}, \quad k=0,1,\ldots,n, \quad n=2,3,\ldots\ .$$

To prove (2.28) it remains to estimate the difference $\theta_k - \theta_k^*$. From (2.6) and (2.40) it follows that

$$\begin{aligned}\left|\sin\frac{\theta_k+\theta_k^*}{2}\sin\frac{\theta_k-\theta_k^*}{2}\right| &= \frac{1}{2}|\cos\theta_k - \cos\theta_k^*| = \frac{1}{2}|\cos\theta_k - a\cos\theta_k| \\ &= \frac{1}{2}(1-a)|\cos\theta_k| \leqslant \frac{1}{2}(1-a) = \frac{1}{2(n^2+1)}. \end{aligned} \tag{2.44}$$

Meanwhile, by virtue of (2.42) and (2.43) we have

$$\begin{aligned}\sin\frac{\theta_k+\theta_k^*}{2} &\geqslant \frac{2}{\pi}\min\left\{\frac{\theta_k+\theta_k^*}{2}, \pi - \frac{\theta_k+\theta_k^*}{2}\right\} \\ &\geqslant \frac{2}{\pi}\min\left\{\frac{\theta_k^*}{2}, \frac{\pi-\theta_k^*}{2}\right\} \geqslant \frac{1}{\pi n}\end{aligned}$$

and

$$\left|\sin\frac{\theta_k-\theta_k^*}{2}\right|\geqslant\frac{|\theta_k-\theta_k^*|}{\pi},$$

which, coupled with (2.44), yield

$$|\theta_k-\theta_k^*|\leqslant\frac{\pi^2 n}{2(n^2+1)}\leqslant\frac{\pi^2}{2n}.$$

At last, we conclude

$$\begin{aligned}\theta_{k+1}-\theta_k&\leqslant\theta_{k+1}^*-\theta_k^*+|\theta_{k+1}^*-\theta_{k+1}|+|\theta_k^*-\theta_k|\\&\leqslant\frac{c\ln(n\mu_n)}{n}+\frac{\pi^2}{n}\leqslant\frac{c\ln(n\mu_n)}{n}.\end{aligned}$$

This completes the proof supposing (2.29). When condition (2.29) does not hold, the statement is obvious. □

Lemma 8. *If relation* (1.5) *is true then*

$$\theta_{k+1,n}-\theta_{kn}\geqslant\frac{c}{n},\quad k=1,2,\dots,n-1,\quad n=2,3,\dots\ . \tag{2.45}$$

Proof. The proof follows the line given by Erdős and Turán [3, p. 718] and [4]. By Rolle theorem and Bernstein inequality

$$\frac{1}{|\theta_{k+1}-\theta_k|}=\left|\frac{A_{0k}(\cos\theta_k)-A_{0k}(\cos\theta_{k+1})}{\theta_k-\theta_{k+1}}\right|=\left|\frac{d[A_{0k}(\cos\theta)]}{d\theta}\right|_{\theta=\theta'}\leqslant c(mn-1),$$

which is equivalent to (2.45). □

3. Proof of theorem

It suffices to show the sufficiency according to Banach–Steinhaus theorem. To this end again applying Banach–Steinhaus theorem it is enough to show that (1.4) holds for every polynomial.

Assume that P is an arbitrary polynomial. Let N be so large that $P\in\mathbf{P}_N$. Then

$$P(x)=\sum_{k=1}^{n}\sum_{j=0}^{m_k-1}P^{(j)}(x_k)A_{jk}(x)$$

and hence

$$R_{nm}(P,x) := |H_{nm}(P,x) - P(x)| = \left|\sum_{k=1}^{n}\sum_{j=1}^{m_k-1} P^{(j)}(x_k)A_{jk}(x)\right|$$
$$= \left|\sum_{k=1}^{n}\sum_{j=1}^{m-1} P^{(j)}(x_k)A_{jk}(x)\right| \leqslant \sum_{j=1}^{m-1}\left|\sum_{k=1}^{n} P^{(j)}(x_k)A_{jk}(x)\right| := \sum_{j=1}^{m-1} S_j. \tag{3.1}$$

(Here we agree that $A_{jk} \equiv 0$ for $j \geqslant m_k$.) To estimate S_j we separate three cases when $j = 1,\ j = 2$, and $j \geqslant 3$ (if $m = 2$ then only the first case can occur).

First, by the same argument as that of [5, Theorem 4.1, p. 73, 74] we can get the estimations

$$S_1 \leqslant 3||P||^* r_{nm}(x) \tag{3.2}$$

and

$$S_2 \leqslant c||P||^* r_{nm}(x), \tag{3.3}$$

where

$$||P||^* := \max_{0 \leqslant j \leqslant m-1} ||P^{(j)}||$$

and

$$r_{nm}(x) := R_{nm}(f_1, x) + R_{nm}(f_2, x).$$

Next, to estimate $S_j\ (j \geqslant 3)$ we need some preliminaries. Relation (4.12) in [5] states that

$$\sum_{k=1}^{n} (x - x_k)^2 A_{0k}(x) \leqslant 2r_{nm}(x),$$

which, together with (2.5), gives

$$\sum_{k=1}^{n} (x - x_k)A_{1k}(x) \leqslant 2r_{nm}(x). \tag{3.4}$$

Using this estimate and applying Lemma B, we obtain estimates

$$||A_{1k}|| \leqslant \frac{8mn||r_{nm}||}{\sin\theta_k}\ (\sin\theta_k \neq 0), \quad k = 1, 2, \ldots, n, \tag{3.5}$$

and

$$||A_{1k}|| \leqslant 4m^2n^2||r_{nm}||, \quad k = 1, 2, \ldots, n. \tag{3.6}$$

Now for the estimate of $S_j\ (j \geqslant 3)$ we use (2.2) and (3.4) to get that for a fixed x,

$$S_j \leqslant ||P^{(j)}|| \sum_{k=1}^{n} |A_{jk}(x)| \leqslant ||P^{(j)}|| \sum_{k=1}^{n} d_k^{j-1}|A_{1k}(x)| \leqslant c||P^{(j)}|| \sum_{k=1}^{n} d_k^2|A_{1k}(x)|$$
$$= c||P^{(j)}|| \left[\sum_{|x_k - x| \geqslant d_k^2} d_k^2|A_{1k}(x)| + \sum_{|x_k - x| < d_k^2} d_k^2|A_{1k}(x)|\right]$$

$$\leqslant c\|P^{(j)}\|\left[\sum_{|x_k-x|\geqslant d_k^2}(x-x_k)A_{1k}(x)+\sum_{|x_k-x|<d_k^2}d_k^2|A_{1k}(x)|\right]$$
$$\leqslant c\|P^{(j)}\|\left[\|r_{nm}\|+\sum_{|x_k-x|<d_k^2}d_k^2|A_{1k}(x)|\right]. \tag{3.7}$$

Here we have to estimate the term $d_k^2|A_{1k}(x)|$. By (1.5) and (2.28),

$$|x_k-x_{k+1}|=|\cos\theta_k-\cos\theta_{k+1}|\leqslant(\theta_{k+1}-\theta_k)(\sin\theta_k+\theta_{k+1}-\theta_k)$$
$$\leqslant c\frac{\ln n}{n}\left(\sin\theta_k+\frac{\ln n}{n}\right);$$

similarly,

$$|x_k-x_{k-1}|=|\cos\theta_k-\cos\theta_{k-1}|\leqslant(\theta_k-\theta_{k-1})(\sin\theta_k+\theta_k-\theta_{k-1})$$
$$\leqslant c\frac{\ln n}{n}\left(\sin\theta_k+\frac{\ln n}{n}\right).$$

Thus

$$d_k\leqslant c\frac{\ln n}{n}\left(\sin\theta_k+\frac{\ln n}{n}\right),\quad k=1,2,\ldots,n. \tag{3.8}$$

We distinguish two cases.

Case 1: $\sin\theta_k\geqslant(\ln n)/n$. In this case by (3.5) and (3.8)

$$d_k^2|A_{1k}(x)|\leqslant c\left(\frac{\ln n}{n}\right)^2\left(\sin\theta_k+\frac{\ln n}{n}\right)^2\frac{8mn\|r_{nm}\|}{\sin\theta_k}$$
$$\leqslant c\left(\frac{\ln n}{n}\right)^2(2\sin\theta_k)^2\frac{8mn\|r_{nm}\|}{\sin\theta_k}$$
$$\leqslant c\frac{(\ln n)^2\|r_{nm}\|}{n}.$$

Case 2: $\sin\theta_k<(\ln n)/n$. In this case by (3.6) and (3.8)

$$d_k^2|A_{1k}(x)|\leqslant c\left(\frac{\ln n}{n}\right)^2\left(\sin\theta_k+\frac{\ln n}{n}\right)^2 4m^2n^2\|r_{nm}\|$$
$$\leqslant c\left(\frac{\ln n}{n}\right)^2\left(2\frac{\ln n}{n}\right)^2 4m^2n^2\|r_{nm}\|$$
$$\leqslant c\frac{(\ln n)^4\|r_{nm}\|}{n^2}$$
$$\leqslant c\frac{(\ln n)^2\|r_{nm}\|}{n}.$$

Thus in both cases

$$d_k^2|A_{1k}(x)| \leqslant c\frac{(\ln n)^2||r_{nm}||}{n}. \tag{3.9}$$

It remains to estimate $K_n := \text{card}\{k: |x_k - x| \leqslant d_k^2\}$. Assume without loss of generality that $\theta_k \leqslant \pi/2,\ 1 \leqslant k \leqslant n-1$. By (2.25) and (2.45)

$$\begin{aligned}|x_k - x_{k+1}| &= |\cos\theta_k - \cos\theta_{k+1}| = |2\sin[(\theta_k + \theta_{k+1})/2]\sin[(\theta_k - \theta_{k+1})/2]| \\ &\geqslant c|(\theta_k + \theta_{k+1})(\theta_k - \theta_{k+1})| \geqslant c(\theta_k - \theta_{k+1})^2 \geqslant c/n^2. \end{aligned} \tag{3.10}$$

Since (3.8) implies $d_k \leqslant c(\ln n)/n$, we can conclude that

$$K_n \leqslant c(\ln n)^2. \tag{3.11}$$

Then by (3.9) and (3.11)

$$\sum_{|x_k - x| \leqslant d_k^2} d_k^2|A_{1k}(x)| \leqslant c\frac{(\ln n)^4||r_{nm}||}{n} \leqslant c||r_{nm}||. \tag{3.12}$$

By (3.1)–(3.3), (3.7), and (3.12) we obtain

$$||R_{nm}(P)|| \leqslant c||P||^*||r_{nm}||. \tag{3.13}$$

Therefore,

$$\lim_{n\to\infty} ||R_{nm}(P)|| = 0.$$

This completes the proof.

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References

[1] P. Borwein, T. Erdélyi, Polynomials and Polynomials Inequalities, Springer, New York, 1995.
[2] P. Erdős, On the uniform distribution of the roots of certain polynomials, Ann. of Math. 43 (1942) 59–64.
[3] P. Erdős, P. Turán, On interpolation II, Ann. of Math. 39 (1938) 703–724.
[4] P. Erdős, P. Turán, On interpolation III, Ann. of Math. 41 (1940) 510–553.
[5] Y.G. Shi, On Hermite interpolation, J. Approx. Theory 105 (2000) 49–86.